Computable error estimates for FEMs for elliptic PDEs with lognormal data

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\textbf{Problem}

As a simple model, consider the homogeneous two-point boundary value problem:

\[-(a(\omega, x) \varphi'(\omega, x))' = 0, \quad \varphi(\omega, 0) = \varphi(\omega, 1) = 0, \quad \varphi(\omega, x) \text{ is a Brownian bridge on } [0, 1], \quad \text{parametrized by each } \omega \in \Omega, \text{ subject to boundary conditions } u(0) = 0 \text{ and } u'(1) = 0, \quad \text{where } \Omega, \mathcal{F}, P \text{ is a probability space. Consider the conductivity } a(\omega, x) = e^{\omega(x)} \text{ where } B \text{ is a Brownian bridge on } [0, 1] \text{ pegged to zero at the end points—a rough lognormal process. One interesting problem is to derive computable estimates for}

\[ E(u - u_h, g), \]

the expected Galerkin error in an observable, where \(u_h\) is the finite element (FE) approximation with mesh size \(h\). In the setting of the rough lognormal conductivity above, the standard \textit{a posteriori} error analysis based on local weighted residuals fails to give reliable estimates.

\textbf{Results}

An assumption on scales yields estimates for the expected Galerkin and quadrature errors committed in standard piecewise linear FE approximations of boundary value problems with rough lognormal conductivities. In particular, these estimates apply to subsurface flow problems in geophysics where the conductivities are assumed to have a lognormal distribution. The computable estimates have the form

\[
\sum_{h\text{-elements}} (\text{local error indicators}) \times h^{d+2}.
\]

The theory is supported by numerical experiments on test problems in one and two dimensions, see [5]. We use the pathwise Galerkin estimate as a tool for obtaining an estimate for the expected Galerkin error in an observable. Below left, we demonstrate the pathwise estimate (\(\star\)) for the generic observable \(g = 1\).

\[ \text{Galerkin error} \quad 2 \times \text{estimate} \]

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Above right, we demonstrate an estimate, based on local error indicators, for a two-dimensional problem where the field \(\log a\) has covariance function \(\text{Cov}(x, y) = e^{-(x-y)^2/2}\). In both cases, using a constant, independent of \(\omega\), produces a reliable estimator. Here the quadrature mesh used in the FE approximation is chosen to overkill the expected error in the setting of a rough lognormal conductivity. For problem \(\star\) for the observable \(g = 1\), the expected quadrature error can also be estimated by local error indicators (figure below).

\[ \text{expected quad. error} \quad 2 \times \text{estimate} \]

Here the calculation is based on \(2^{15}\) samples to ensure that the statistical error is negligible and the one-sided error bars indicate 5 deviations from the mean.

\textbf{Some background}

For Monte Carlo-type methods for this problem, research has mainly focused on questions of well-posedness, convergence rates, and \textit{a priori} error estimates Galerkin (see [1–3, 6] and the references therein).

\[ \sum_{h\text{-elements}} \frac{h^3}{16} a_h^2 D_{u_h}^2 D^2 \hat{h}_h^2 \]

where \(a_h^2\) is the harmonic mean of \(a\) over \(h\)-elements, \(\hat{h}_h^2\) is the discrete dual (depending on \(h\)), and \(D^2\) is the second order central difference.

\textbf{Methods}

In contrast to the case of a smooth conductivity, for a rough lognormal \(a\) the components of the dual weighted residual Galerkin error density contain \textit{non-negligible high-frequency content} that cannot be computed directly. Analyzing the frequency content of these components for the simple model \(\star\) suggests that this high-frequency contribution can be approximated by low-frequency content. The computable estimates are then obtained by making an assumption on scales of the model problem (related to decay of the frequency content) and a simple telescoping argument. Similar arguments are used to obtain the estimate for the quadrature error.

\textbf{Future work}

Estimates of this form are useful for constructing \textit{adaptive algorithms}. We intend to analyze variance reduction techniques for Monte Carlo methods (e.g., MLMC) where the final level stopping criterion is built upon the derived error estimates. Using these estimates, it is also possible to implement adaptive FEMs, in a goal-oriented framework, for elliptic PDEs with rough coefficients.

\textbf{References}
